PORTUGALIAE MATHEMATICA

VOLUME 26

1967

Edição de

«GAZETA DE MATEMÁTICA, LDA.»

PORTUGALIAE MATHEMATICA
Rua Diário de Notícias, 134, 1.º-Esq.
LISBOA-2 (PORTUGAL)

PORTUGALIAE MATHEMATICA
Vol. 26 — Fasc. 4 — 1967

AN IMBEDDING SPACE FOR SCHWARTZ DISTRIBUTION ON R^n (*)

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Introduction.

Springing from Heavisides operational calculus, the first use of generalized functions was intimately connected with the LAPLACE transform. It was not until L. SCHWARTZ's «Theorie des Distributions» [6] had appeared that a rigorous theory was constructed for generalized functions or distributions as the elements of a dual space. The operational calculus was however largely omitted from this work. B. VAN DER POL [9] compiled a most complete work on the operational calculus, but not in the context of distributions and MIKUSINSKI [2] updated this in much of the distribution point of view. The difficulty of using LAPLACE transforms remained that of existence of the transform even if the unilateral transform was used and further the difficulty of inversion. This author [4] succeeded in using the bilateral transform for functions on R^1 to construct a space sufficiently large to imbed the SCHWARTZ Space. In this paper we extend the results of that paper to R^n and obtain a rather complete characterization of this imbedding space.

We proceed in a manner analogous to that in [4]. $\{e^{-z \cdot t}\}$ is a one parameter family of infinitely differentiable functions with respect to $t \in R^n$ where z is in a polycylinder S in C^n, and an L_S-Distribution is defined to be a functional on this family that is analytic with respect to the parameter and further we require that if α is a complex scalar and F, G two such

* Received October, 1966.
(1) This research was partially supported by National Science Grant GP1591.

functionals then $F \cdot e^{-s \cdot t} + G \cdot e^{-s \cdot t} = (F+G) \cdot e^{-s \cdot t}$ and $(\alpha F) \cdot e^{-s \cdot t} = F \cdot (\alpha e^{-s \cdot t})$. In a manner analogous to that for SCHWARTZ Distributions the derivative of such a functional is defined by $F' \cdot e^{-s \cdot t} = F \cdot (e^{-s \cdot t})'$. For a properly defined LAPLACE Transform it is clear that the transform provides an integral representation for some L_S-Distributions and in turn an ordinary analytic function representation. To construct the imbedding space we begin with the space of functions on C^n analytic in a polycylinder $S \subset C^n$. Following the terminology used in [4] and by other authors we shall, when convenient, refer to the inverse LAPLACE Transform, symbolic or not, as an L_S-Distribution rather than the functional.

2. Notation and Conventions.

C denotes the complex plane and $C^n = C \times \cdots \times C$ with the usual product topology. $z \in C^n$ means $z = (z_1, \cdots, z_n)$, $z_j = x_j + iy_j \in C$. R denotes the real line and $R^n = R \times \cdots \times R$, $t \in R^n$ means $t = (t_1, \cdots, t_n), t_j \in R$. For $z, w \in C^n$ define

$$z \cdot w = \sum_{j=1}^{n} z_j \cdot \overline{w}_j = \overline{w} \cdot \overline{z}$$

$\overline{w}_j$ being the complex conjugate. Two different absolute value functions will be used, for $z \in C^n$ or $t \in R^n$, $(|z|) = (|z_1|, \cdots, |z_n|)$ and $|z| = \sqrt{z \cdot z}$ and further $R(z) = (R(z_1), \cdots, R(z_n))$.

With the usual topology, R^n is a HILBERT space and if $\overline{\eta}$ is in the dual then there is an $\eta \in R^n$ such that $\overline{\eta}(t) = \eta \cdot t$ for all $t \in R^n$. For any two such functionals $\overline{\eta}$ and $\overline{\tau}$ where $\eta < \tau$, let

$$S(\overline{\eta}, \overline{\tau}) = \{ z \mid \eta < R(z) < \tau \}.$$

We also have

$$S = \prod_{j=1}^{n} \{ z_j \mid \eta_j < R(z_j) < \tau_j \}.$$

3. Ls-D Space-Topologies.

DEFINITION 3. 1. Let $\bar{\eta}$ and $\bar{\tau}$ be in the dual of R^n such that $\eta < \tau$ and $S = \{z \mid \eta < R(z) < \tau\}$ then if $f(z)$ is analytic in S, $[f(z)]_t$ is defined to be an Ls-Distribution whose value at $e^{-z \cdot t}$ is $f(z)$. Further $f(z)$ is the LAPLACE Transform of $[f(z)]_t$.

DEFINITION 3. 2. Let $D^\alpha = \dfrac{\partial^{\alpha_1} \partial^{\alpha_2} \partial^{\alpha_3} \partial^{\alpha_n}}{\partial t_1^{\alpha_1} \cdots \partial t_n^{\alpha_n}}$ then $D^\alpha f_t$ is defined to be $[z_1^{\alpha_1} \cdots z_n^{\alpha_n} f(z)]_t$ where $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $z = (z_1, \cdots, z_n)$.

DEFINITION 3. 3. For an arbitrary S as in Definition 1, let S_L be the space of Ls-Distributions with the topology induced by uniform convergence on compact subset of S.

It is clear that for any S, S_L is a complete linear space closed with respect to the operator D^α for all α. The following theorems justify in part inducing the topology on the space of Ls-D's from the space of analytic functions. In particular we show that when the distribution actually is determined by a function on R^n as the transform of that function then the induced topology is compatible with a topology that might be defined on the space of such functions on R^n

For $\eta < \tau$ let $L_2(\bar{\eta}, \bar{\tau})$ denote the space of functions on R^n such that

$$\| F \|_j = \left[(-1)^{b_j} \int_{R_j^n} | e^{-{}_j\sigma \cdot t} F(t) |^2 dt \right]^{1/2} < \infty$$

where ${}_j\sigma = (\sigma_1, \cdots, \sigma_n)$ and each σ_i is either η_i or τ_i and $R_j^n [0, a_1) \times \cdots \times [0, a_n)$ with $a_i = +\infty$ or $-\infty$ correspondingly, and b_j is the number of a_i's that are $-\infty$, $j = 1, 2, \cdots, 2^n$. Let

$$\| F \| = \sum_{j=1}^{2^n} \| F \|_j .$$

THEOREM 3. 4. $L_2(\overline{\eta}, \overline{\tau})$ *is a complete linear space, each* F *in* L_2 *has a Laplace Transform,* $f(z)$, *which is analytic for* z *in* $S(\overline{\eta}, \overline{\tau})$. *Further convergence in* $L_2(\overline{\eta}, \overline{\tau})$ *implies uniform convergence of the transforms on compact subsets of* R.

PROOF. From the identity

$$\int_{R^n} e^{-\xi \cdot t} F(t)\, dt = \sum_{j=1}^{2^n} (-1)^{b_j} \int_{R_j^n} e^{-\xi \cdot t} F(t)\, dt$$

for any $\xi = (\xi_1, \cdots, \xi_n)$ such that both sides exist and the inequalities $e^{-\xi \cdot t} \leqq e^{-{}_j\sigma \cdot t}$ for $t \in R_j^n$ and $\eta \leqq \xi \leqq \tau$ it follows that $f(z) = \int_{R^n} e^{-z \cdot t} F(t)\, dt$ converges absolutely and is analytic for $z \in S(\overline{\eta}, \overline{\tau})$ by iterated application of the existence theorem for $n = 1$ (See pp. 245 [8]). It is clear that $\| \ \|$ is a norm and that L_2 is a linear space. The completeness of $L_2(\overline{\eta}, \overline{\tau})$ follows from the completeness of $L_2(\overline{\eta}, \overline{\eta})$ and of $L_2(\overline{\tau}, \overline{\tau})$ and the uniqueness of the limits. To prove uniform convergence write

$$f(z) = \int_{R^n} e^{-z \cdot t} F(t)\, dt = \sum_{j=1}^{2^n} (-1)^{b_j} \int_{R^n} e^{-z \cdot t} F(t)\, dt$$

and

$$\int_{R_j^n} e^{-z \cdot t} F(t)\, dt = \int_{R_j^n} e^{-(z - {}_j\sigma) \cdot t} e^{-{}_j\sigma \cdot t} F(t)\, dt.$$

Then by n applications of the CAUCHY-SCHWARTZ Inequality

$$\left| \int_{R_j^n} e^{-z \cdot t} F(t)\, dt \right| \leqq [g_j(z)] \left[\int_{R_j^n} | e^{-{}_j\sigma \cdot t} F(t) |^2\, dt \right]^{1/2}$$

where $g_j(z) = \prod_{i=1}^{n} [2 \,|\, R(z_i) - {}_j\tau_i \,|]^{-1/2}$. Then

$$|f(z)| \leqq \max_{1 < j < n} |g_j(z)| \, \| F \|$$

and hence if $\| F_k \| \to 0$ as $k \to \infty$ then $|f_k(z)| \to 0$ uniformly on compact subset of S as $k \to \infty$.

COROLLARY 3. 4. *Let* $f_m(z)$ *be the transform of functions* $F_m(t)$ *in* $L_2(\bar{\eta}, \bar{\tau})$, *then* $\| f_m \|_x \to 0$ *as* $\| F_m \| \to 0$ *as* $m \to \infty$ *where* $\| f \|_x = \left[\int_{R^n} f(x+iy)|^2 dy \right]^{1/2}$ *and* $\eta < x < \tau$.

PROOF. By iterated application of theorem for $n=1$ (pp 245, [8])

$$\int_{R^n} |e^{-x \cdot t} F(t)|^2 dt = \int_{R^n} |f(x+iy)|^2 dy$$

for $F(t) \in L_2(\bar{\eta}, \bar{\tau})$, $f(z)$ the transform of F and $\eta < x < \tau$. Then since

$$(-1)^{b_j} \int_{R_j^n} |e^{-x \cdot t} F(t)|^2 dt \leq [\| F \|_j]^2$$

the results follows.

4. Ls-Distributions.

DEFINITION 4. 1. For any Ls-D, F_t, define f_{t+h} to be $[e^{h \cdot z} f(z)]_t$ where $h = (0, \cdots, h_j, 0_j, \cdots 0)$.

THEOREM 4. 2. *For any* Ls-D, f_t, $\frac{1}{h_j}[f_{t+h} - f_t] - D^\alpha f_t \to 0$ as $h_j \to 0$ *where* $\alpha = (0, \cdots, 1, \cdots, 0)$.

PROOF. By definition $D^\alpha f_t = [z_j f(z)]_t$ hence

$$\frac{1}{h_j}[f_{t+h} - f_t] - D^\alpha f_t = \left[f(z) \frac{(e^{h_j z_j} - 1 - z_j)}{h_j} \right]_t$$

and since $f(z)$ is bounded on compact sets and

$$\frac{e^{h_j z_j} - 1 - z}{h_j} = h_j \left[\frac{z_j^2}{2!} + \frac{h_j z_j^3}{3!} + \cdots \right]$$

converges uniformly to 0 on compact sets as $h_j \to 0$ the proof is camplete

THEOREM 4.3. *For any sequence* $\{{}_m f_t\}$, *convergent to* f_t, *of* Ls-D's *and arbitrary* D^α, $D^\alpha_m f_t$ *converges to* $D^\alpha f_t$.

PROOF. By definition ${}_m f_t \to f_t$ as $m \to \infty$ if ${}_m f(z) - f(z) \to 0$ uniformly on compact sets. Since D^α is linear

$$D^\alpha [{}_m f(z)]_t = D^\alpha [f_m(z)] - D^\alpha [f(z)]_t$$
$$= \left[\prod_{j=1}^{n} z_j^{\alpha_j} ({}_m f(z) - f(z)) \right]_t .$$

For any α, $\prod_{j=1}^{n} z_j^{\alpha_j}$ is bounded on compact sets so that

$$\prod_{j=1}^{n} z_j^{\alpha_j} ({}_m f(z) - f(z) \to (0)$$

uniformly on compact sets if

$${}_m f(z) - f(z)) \to 0 .$$

DEFINITION 4.4. For any pair of Ls-D's f_t and g_t, $f_t^* g_t = [f(z) g(z)]_t$ is called the Convolution Product.

We note that if f_t and g_t are determined by the LAPLACE Transforms of $F(t)$ and $G(t)$, respectively, then by the Convolution Theorem for such transforms $f(z)\, g(z)$ is the transform of

$$\int_{R^n} F(t-u)\, G(u)\, du .$$

The definition of the convolution product is then consistent with the usual definition for point functions. We also easily obtain the following properties of this product.

THEOREM 4.5. *For any pair of* Ls-D's f_t *and* g_t *and any differential operator* D^α

(i) $f_t * g_t = g_t * f_t$

(ii) $D^\alpha (f_t * g_t) = (D^\alpha f_t) * g_t = f_t * (D^\alpha g_t)$.

PROOF (*i*). By definition

$$\begin{aligned} f_t * g_t &= [f(z)g(z)]_t \\ &= [g(z)f(z)]_t \\ &= g_t * f_t \end{aligned}$$

(*ii*) Likewise, by definition

$$\begin{aligned} \mathrm{D}^\alpha (f_t * g_t) &= \left[\prod_{j=1}^{n} z_j^{\alpha_j} (f(z)\, g(z)) \right]_t \\ &= \left[\left(\left(\prod_{j=1}^{n} z_j^{\alpha_j} \right) f(z) \right) g(z) \right]_t \\ &= (\mathrm{D}^\alpha f_t) * g_t \end{aligned}$$

and the remainder follows from (*i*) applied to obtain

$$\mathrm{D}^\alpha (f_t * g_t) = \mathrm{D}^\alpha (g_t * f_t).$$

THEOREM 4. 6. *If* ${}_m\mathrm{f}_t \to 0$, *then for any* g_t, ${}_m\mathrm{f}_t * \mathrm{g}_t \to 0$

PROOF. By definition ${}_mf_t \to 0$ if and only if ${}_mf(z) \to 0$ uniformly on compact sets. But for fixed $g(z)$, the ${}_mf(z)\,g(z) \to 0$ uniformly on compact sets, hence

$${}_mf_t * g_t \to 0.$$

A second product can be defined for those pairs of $\mathrm{L}_S - \mathrm{D}'s$ that are separately determined by polycylinders S_{n_1} and S_{n_2} where $\mathrm{S} = \mathrm{S}_{n_1} \times \mathrm{S}_{n_2}$, That is, f_v is determined by $f(\zeta_1)$ and g_u by $g(\zeta_2)$ with $v \in \mathrm{R}^{n_1}, u \in \mathrm{R}^{n_2}, \mathrm{R}^{n_1} \times \mathrm{R}^{n_2} = \mathrm{R}^n$.

THEOREM 4. 7. *For* f_v on R^{n_1} and g_u on R^{n_2} *there exists a unique* $\mathrm{h}_t, t \in \mathrm{R}^n$ *such that* $\mathrm{h}_t = [\mathrm{h}(\mathrm{z})]_t$ *and* $\mathrm{h}(\mathrm{z}) = \mathrm{f}(\zeta_1)\,\mathrm{g}(\zeta_2)$ *where* $\mathrm{z} = (\zeta_1, \zeta_2)$.

PROOF. Without loss of generality we can assume $z = (z_1, \ldots, z_n)$ and $\zeta_1 = (z_1, \cdots, z_{n_1})$, $\zeta_2 = (z_{n_1}, \ldots, z_n)$ since

$n_1+n_2=n$. It is clear that $h(z)$ is uniquely determined by f and g and $h(g)$ is analytic in $z \varepsilon S$.

We denote this product by $h_t = f_v \otimes g_u$, called the tensor product. By way of consistency we see that if $F(v)$, $G(u)$ possess transforms $f(\zeta_1)$ and $g(\zeta_2), v \varepsilon R^{n_1}, u \varepsilon R^{n_2}$ then

$$h(z) = \int_{R^n} e^{-z \cdot (t_1+t_2)} F(v) G(u) d(t_1+t_2)$$

where $t_1 = (v, 0) \varepsilon R^n, t_2 = (0, u) \varepsilon R^n$, becomes

$$\left(\int_{R^{n_1}} e^{-\zeta_1 \cdot v} F(v) dv\right) \left(\int_{R^{n_2}} e^{-\zeta_2 \cdot u} G(u) du\right) = f(\zeta_1) g(\zeta_2).$$

Corollary 4. 7. 1. *Theorems 4. 5. and 4. 6. are also true for the tensor product.*

Proof. This essentially is the same as for those theorems.

At this point we make some general observations about the space of $L_S - D's$. First it contains isomorphically those functions on R^n which possess transforms, invariant with respect to differentiation and convolution where appropriate and with compatible topologies. Included in the former category are the elements of the group algebra $L^1(R^n)$ of R_n as a locally compact abelian group and a subset of the space of infinity differentiable functions on R^n with compact support. It is well known that $L^1(R^n)$ does not have an identity element but by using the entire ring of functions analytic in S we have adjoined an identity element and also have the advantage of a complete, separable space under the topology of uniform convergenc on compact sets. At the same time however, we see that we do have sufficiently many «generalized functions». In particular the Schwartz Distributions represented locally via the Riesz Representation Theorem in terms of a constant function are not present (see [1] or [6]). The construction of a «larger» space is proceeded with in the next section. By contrast with the Schwartz construction: the consistency of the definition of differentiation, the continuity of the differential operator and both products are rather easy consequences of their definitions.

Finally we note that if we follow a construction analogous to that used by MILLER [3], then the space is S_L. That is, let

$$\mathfrak{X} = \{F \mid F \varepsilon L^1(R^n), f(z) = \int_{R^n} e^{-z\cdot t} F(t)\, dt \text{ exists for som } z \varepsilon C\}.$$

For any compact subset $\mathcal{C} \subset C$, let

$$\mathfrak{X}_{\mathcal{C}} = \{F \mid F \varepsilon \mathfrak{X}, \text{ for } f(z) \text{ exists for all } z \varepsilon \mathcal{C}\}$$

and define $\| F \|_{\mathcal{C}} = \max_{z \in \mathcal{C}} |f(z)|$, finally denote by $\overline{\mathfrak{X}}_{\mathcal{C}}$ the completion of $\mathfrak{X}_{\mathcal{C}}$. To continue

$$\Pi_x = \{z \mid -x < R(z) < x, |z| \neq \infty\}$$
$$C_x = \{\mathcal{C} \mid \mathcal{C} \subset \Pi_x, \bigcup \mathcal{C}^0 \supset \Pi_x,$$
$$\mathcal{C} = \bigcup {}_j\mathfrak{A}_{\mathcal{C}}, {}_j\mathfrak{A}_{\mathcal{C}} \cap {}_k\mathfrak{A}_{\mathcal{C}} = \emptyset,$$
$${}_j\mathfrak{A}_{\mathcal{C}} \text{ is a Jordan Region}\}.$$

If now for each $\mathcal{C} \in C_x$, $\mathcal{B}(\mathcal{C})$ is a β-algebra of functions in $\overline{\mathfrak{X}}_{\mathcal{C}}$ then construct a projective spectrum (as defined by S. E. SILVA [7]). If $\mathcal{B}_x$ is the projective limit then $\mathcal{B}_x$ is homemorphic with the space of all functions analytic in Π_x. Consequently we are still led to consider the space of functions analytic in a strip S. In a subsequent paper we expect to study such a β-algebra in more detail. For fixed $\eta, \tau \in R^n$ let L^2_S be the set of functions $F(t)$ such that $e^{-\sigma\cdot t} F(t)$ is in L^2 for $\eta < \sigma < \tau$. It is clear that L^2_S is a linear space, with convolution as a product and by Theorem 3.4 each has a transform with the Plancherel Theorem relating the norms of $F \in L^2_S$ and the mean-square norms of their transforms.

5. Generalized L_S-Distributions.

As noted above, the space constructed is not sufficiently rich to contain all the generalized functions which would correspond to constant functions. We now proceed to remedy this.

DEFINITION 5.1. An Ls-D, f_t, is said to have point-values $F(t)$ for $c<t<d$ if there exists a $G(t)$ such that for some $\alpha=(\alpha_1, \cdots, \alpha_n)$,

$$f(z)=\left(\prod_{j=1}^{n} z_j^{\alpha_j}\right) g(z)$$

$$D^{\alpha} G(t)=F(t), \quad c<t<d$$

where $g(z)$ is the Laplace Transform of $G(t)$.

For example $[1]_t$ has zero-point value for all $t=(t_1, \cdots, t^n)$, $t_j \neq 0$. Since

$$\begin{aligned} H(t) = 1 \quad & t>0 \\ 0 \quad & \text{otherwise} \end{aligned}$$

has as its transform

$$h(z)=\left(\prod_{j=1}^{n} z_j\right)^{-1}$$

and $[1]_t=\left[\left(\prod_{j=1}^{n} z_j\right) h(z)\right]_t$ with $\dfrac{\partial^n H(t)}{\partial t_1, \cdots, \partial t_n}=0$ if $t=(t_1, \cdots, t_n)$, $t_j \neq 0$.

$[1]_t$ is the «Dirac Delta Function» and the identity element for the tensor product.

DEFINITION 5.2. A sequence, $\{{}_m f_t\}$ of Ls-D's is called fundamental if for each open set, $c<t<d$, there exists an integer M such that for $m>M$,

$${}_m f_t - {}_{m+p} f_t, \quad p=0, 1, 2, \cdots$$

is an Ls-D with zero point values in (c, d). A fundamental sequence of Ls-D's will be abreviated F.S.S.

DEFINITION 5.3. Two F.S.S.'s, $\{_m g_t\}$ and $\{_m f_t\}$, are said to be similar if for each open set, $c < t < d$, there exists an integer M such that for $m > M$

$$_m g_t - _m f_t$$

is an L_S-D with zero point-values in (c, d).

LEMMA 5.4. *The similarity defined in* DEFINITION 5.3 *is a Linear Equivalence relation and is invariant under differentiation, and both products.*

The proof is an immediate consequence of the definitions and will be omitted.

THEOREM 5.5. *The Equivalence classes under the similarity relation are called Generalized* L_S-D's *or* G · L_S-D's. *The* G · L_S-D's *form a linear space over the complex field with continuous differentiation and with Convolution and Tensor products.*

THEOREM 5.6. *Let* S *be a non-degenerate strip in the complex* n-*plane*, C^n. *Then there is a subspace*, $\mathfrak{X}$, *of the space of* G · L_S-D's *such that* $\mathfrak{X}$ *is isomorphic with the* SCHWARTZ *space* (D'). *This isomorphism is invariant with respect to addition, scalar multiplication, differentiation convolution and tensor multiplication.*

The proof is in three parts.

(*a*) By definition a SCHWARTZ Distribution is a linear functional on the space (D) of C^∞ functions with compact supports and is continuous on the elements of C^∞_A for fixed A compact. Let $C^\infty[a_m, b_m]$ denote the set of C^∞ functions on R^n whose support is contained in $[a_m, b_m]$. Then for any $\phi \in C^\infty[a_m, b_m]$ the value of the distribution T at ϕ is given by

$$T(\phi) = (-1)^{|\alpha|} \int_{[a_m, b_m]} F(t) D^\alpha \phi(t) dt$$

where $F(t)$ is continuous on $[a_m, b_m]$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and α is determined by T and $[a_m, b_m]$. Suppose now that $m \to \infty$

such that $-\infty \to a_{m+1} \leqq a_m < b_m \geqq b_{m+1} \to \infty$. For each m there is an $F_m(t)$ and an ${}_m\alpha$.

$$G_m(t) = (-1)^{|{}_n\alpha|} F_m(t), \; {}_m a \leqq t \leqq {}_m b$$
$$= 0 \quad \text{otherwise}$$

$${}_m f(z) = \int_{[a_m, b_m]} e^{-z.t} \prod_{j=1}^{n} z_j^{\alpha_j} G_m(t)\, dt.$$

We now need to show that $\{{}_m f_t\}$ is an F.S.S. and uniquely determines a G · L_S-D. independent of the expanding sequence of rectangles $[a_m, b_m]$. This G · L_S-D. is the representative of T.

(*b*) Let I be an arbitrary open set (c, d). There exists an integer M such that for $m > M$, $[a_m, b_m] \supset (c, d)$. Let $F_m(t)$, $F_{m+p}(t)$, ${}_m\alpha$, ${}_{m+p}\alpha$ be the continuous functions and n-tuples of integers for the representation of T on the intervals $[a_m, b_m]$, $[a_{m+p}, b_{m+p}]$ respectively. Since $C^\infty[a_m, b_m] \subset C^\infty[a_{m+p}, b_{m+p}]$, for $\phi \in C^\infty[a_m, b_m]$,

$$T(\phi) = (-1)^{|{}_m\alpha|} \int_{[a_m, b_m]} F_m(t) D^{{}_m\alpha} \phi(t)\, dt$$
$$= (-1)^{|{}_{m+p}\alpha|} \int_{[a_{m+p}, b_{m+p}]} F_{m+p}(t) D^{{}_{m+p}\alpha} \phi(t)\, dt$$

or

$$\int_{[a_m, b_m]} [F_m(t) D^{{}_m\alpha} \phi(t) - (-1)^{|{}_{m+p}\alpha| - |{}_m\alpha|} F_{m+p}(t) D^{{}_{m+p}\alpha} \phi(t)]\, dt.$$

For $\alpha = (0, \cdots, 1, 0, \cdots, 0)$ an n-tuple with all but one α_j zeros and that $\alpha_j = 1$, let

$$D_a^{-\alpha} F(t) = \int_{a_j}^{t_j} F(x_j)\, dx_j$$

and in general for α an n-tuple of non-negative integers, define $D^{-\alpha}$ interatively. For each pair of n-tuples ${}_m\alpha$, ${}_{m+p}\alpha$ let ${}_p\alpha^+$

$=(\gamma_1,\cdots,\gamma_n)$, ${}_p\alpha^-=(\delta_1,\cdots,\delta_n)$ where $\gamma_j=\max(0,{}_{m+p}\alpha_j-{}_m\alpha_j)$, $\delta_j=\max(0,{}_m\alpha_j-{}_{m+p}\alpha_j)$. It follows that

$$0=\int_{[{}_ma,{}_mb]}[(-1)^{|{}_{m+p}\alpha|-|{}_m\alpha|}D^{-{}_p\alpha^+}F_m(t)-D^{-{}_p\alpha^-}F^{m+p}(t)]$$

$$\cdot D^{{}_{m+p}\alpha}\phi(t)\,dt=\int_{[{}_ma,{}_mb]}Q_\beta(t)D^{{}_{m+p}\alpha}\phi(t)\,dt.$$

It is clear that since this is true for al $\phi\in C^\infty[{}_ma,{}_mb]$, $Q_\beta(t)$ is a polynomial in t, of degree $\beta\leqq{}_{m+p}\alpha-(1,\cdots,1)$, for ${}_ma\leqq t\leqq{}_mb$. We may now write

$${}_{m+p}f(z)-{}_mf(z)=\int_{[{}_ma,{}_mb]}e^{-z\cdot t}\left(\prod_{j=1}^n z_j^{{}_{m+p}\alpha_j}\right)Q_\beta(t)\,dt$$

$$+\int_{[{}_{m+p}a,{}_{m+p}b]-[{}_ma,{}_mb]}e^{-z\cdot t}\left(\prod_{j=1}^n z_j^{{}_{m+p}\alpha_j}\right)G^{m+p}(t)\,dt.$$

The first integral is the transform of the derivative of a function with zero point values interior to and exterior to the open interval $[{}_ma,{}_mb]$ and hence to the interval (c,d). The second integral is the transform of the derivative of a function with zero-point values in the exterior of $[{}_{m+p}a,{}_{m+p}b]-[{}_ma,{}_mb]$. HENCE the constructed sequence $\{{}_mf_t\}$ is an F. S. S.

(c) Suppose $[{}_ma,{}_mb]$, $[{}_mc,{}_md]$ are two expanding sequences whose unions cover R_n. Let $({}_mf_t)$ and $({}_mg_t)$ be the F. S. S.'s constructed as in part (a). Let I be an arbitrary bounded open set in R^n. There exisie integers M_I^1, M_I^2 such that ${}_{m+p}f_t-{}_mf_t$ is an L_S-D with zero point-valuds in I for $m>M_I^1$ and ${}_{m+p}g-{}_mg_t$ is an L_S-D with zero point-values in I for $m>M_I^2$. Further, there exists an integer M such that $[{}_ma,{}_mb]\subset[N^c,N^d]$ for $N=M_I^2, m\geqq M$. If $K=\max(M_I^1,M_I^2)$ the write.

$${}_mf_t-{}_mg_t=[{}_mf_t-{}_kf_t]+[{}_kf_t-{}_kg_t]+[{}_kg_t={}_mg_t].$$

For $m>K$, the first difference on the right is an L_S-D with zero point-values in I since $\{{}_mf_t\}$ is an F. S. S. the second difference can be shown to be an L_S-D with zero point-values in I by the method in part (*b*). The third difference is an L_S-D with zero point-values in I since $\{{}_m g_t\}$ is an F. S. S. We have now established that the correspondence between SCHWARTZ Distributions and G · L_S-D's is one-to-one. The invariance properties follow from Lemma 5.4.

6. Topologizing the Imbedding Space.

DEFINITION 6.1. An F. S. S. $\{{}_m f_t\}$ is said to have point-values $F(t)$ in an interval (c, d) if there exists an integer M such that for $m > M$, ${}_m f_t$ is an L_S-D with point values $F(t)$ in (c, d). A G · L_S-D is said to have point-values $F(t)$ in an interval (c, d) if there exists an F. S. S., in the equivalence class of the G · L_S-D, possessing that property.

DEFINITION 6.2. Let $\{{}_m f_t\}_1, \ldots, \{{}_m t_t\}_k, \ldots$ be a sequence of F. S. S.'s. Denote the m^{th} element of the k^{th} F. S. S. $({}_m f_t)_k$. Then the sequence of F. S. S.'s is said to converge to the sequence L_S-D's $\{{}_m f_t\}_0$ if ${}_m f(z)_k \to ({}_m f(z))_0$ uniformly on compact sets as $m, k \to \infty$.

DEFINITION 6.3. Let $D_1, \cdots, D_j, \cdots$ be a sequence of G·L_S-D·'s. Further, suppose $L_1, L_2, \cdots$ is a sequence of F. S. S.'s each having zero point-values exterior to $[a, b]$ and for each j, L_j represents D_j on $[a, b]$. That is, for some F. S. S., $\{{}_m f_t\}_j$, in D_j, $L_j - \{{}_m f_t\}_j$ has zero point-values in (a, b). Then if $L_1, \cdots, L_j, \cdots$ is convergent to L_0 in the sense of *Definition* 6.2, $D_1, \cdots, D_j$ is said to converge to D_0 on (a, b), D_0 being determined by L_0.

THEOREM 6.4. *If a sequence of* SCHWARTZ *Distributions is convergent in the open interval* (a, b) *in the* SCHWARTZ *space* (D′), *then the sequence of* G · L_S-D's *isomorphic to the respective* SCHWARTZ *Distributions is convergent in the interior of every closed sub-interval of* (a, b).

PROOF. Let $T_1, \cdots, T_j, \cdots,$ be a sequence of SCHWARTZ Distributions convergent in (D′) in the interval (a, b). For any closed sub-interval $[c, d]$ of (a, b) there exists a sequence of representations.

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